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Robb, Alfred Arthur

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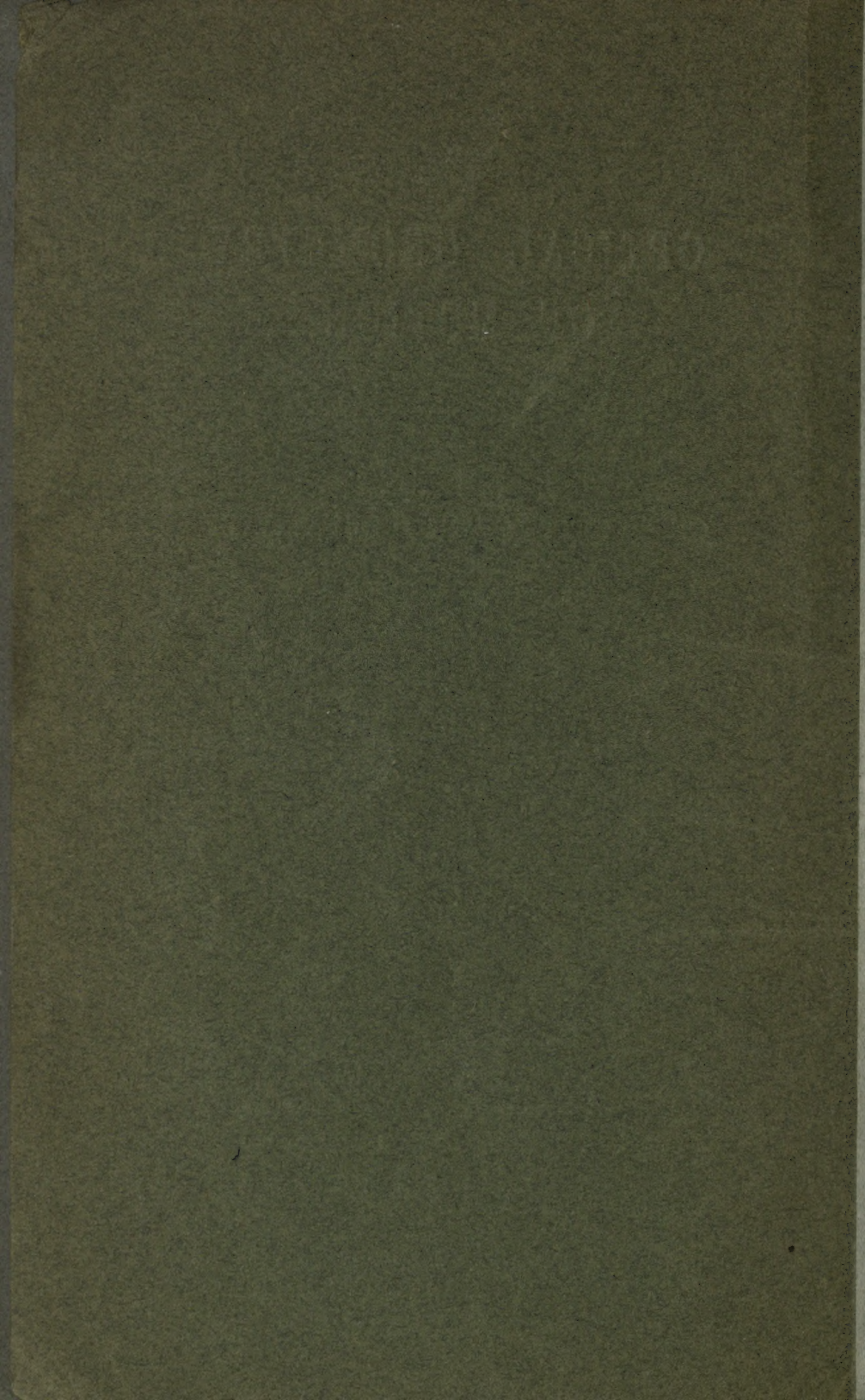
OPTICAL GEOMETRY
OF MOTION

A NEW VIEW
OF THE THEORY OF RELATIVITY

BY
ALFRED A. ROBB, M.A., PH.D.

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PREFACE.

IN placing before his readers the following brief outline of a point of view, the writer is well aware that it is far from complete in many respects. He however believes that, in the first presentment of a theory, there are considerable advantages in stating explicitly only its principal features.

To cover up a general standpoint under a mass of detail is to run the risk of obscuring it altogether. There is always the danger that the reader may "not be able to see the wood for the trees"—a danger which is becoming very real in much modern mathematics.

The substance of the following essay was originally intended by the writer to form a chapter of a book of semi-philosophical character upon which he is engaged.

In view, however, of the amount of attention which the subject of Relativity is at present attracting, it seemed to him that this portion might prove of sufficient interest to warrant its separate publication.

From the standpoint of the pure mathematician *Geometry* is a branch of *formal logic*, but there are more aspects of things than one, and the geometrician has but to look at the name of his science to be reminded that it had its origin in a definite *physical* problem.

That problem in an extended form still retains its interest.

A. A. ROBB.

Cambridge,
May 13th, 1911.

“But deepest of all illusory Appearances, for hiding Wonder, as for many other ends, are your two grand fundamental world-enveloping Appearances, SPACE and TIME. These, as spun and woven for us from before Birth itself to clothe our celestial ME for dwelling here, and yet to blind it,—lie all-embracing, as the universal canvas, or warp and woof, whereby all minor Illusions in this Phantasm Existence weave and paint themselves. In vain, while here on Earth shall you endeavour to strip them off; you can, at best, but rend them asunder for moments, and look through.”

CARLYLE, “*Sartor Resartus*.”

OPTICAL GEOMETRY OF MOTION.

Introduction.

THE foundations of Geometry have been carefully investigated, especially of late years, by many eminent mathematicians. These investigations have (with the notable exception of those of Helmholtz) been almost all directed towards the Logical aspects of the subject, while the Physical standpoint has received comparatively little attention.

Speaking of the different "Geometries" which have been devised, Poincaré has gone so far as to say that: "one Geometry cannot be more true than another; it can only be more convenient." In order to support this view it is pointed out that it is possible to construct a sort of dictionary by means of which we may pass from theorems in Euclidian Geometry to corresponding theorems in the Geometries of Lobatschewskij or Riemann.

In reply to this; it must be remembered that the language of Geometry has a certain fairly well defined physical signification which *in its essential features* must be preserved if we are to avoid confusion.

As regards the "dictionary," we would venture to add that it would also be possible to construct one in which the ordinary uses of the words *black* and *white* were interchanged, but, in spite of this, the substitution of the word *white* for the word *black* is frequently taken as the very type of a falsehood.

It is the contention of the writer that the axioms of Geometry, with a few exceptions, may be regarded as the formal expression of certain Optical facts. The exceptions are a few axioms whose basis appears to be Logical rather than Physical.

It is proposed in the following pages to refer briefly, in the first place, to certain Optical phenomena which occur in free space, upon which we might suppose some of the

chief axioms to have their foundations; and then to employ these to establish on a new basis some of the groundwork of the theory of "Relativity."

The writer does not propose, in the present paper, to go into the more minute Logical details of the foundations of Geometry; as it seems to him that these would tend to obscure the general standpoint which he desires to emphasize. For this reason he prefers to reserve them for a future occasion.

OPTICAL GEOMETRY OF REST.

In the application of ordinary Geometry as distinguished from Kinematics, we are concerned with systems which preserve their configuration unchanged. We shall first consider briefly such systems. Our sense of vision supplies us with a direct means by which we can tell that a particle in free space lies in the same straight line as two other particles. If three particles A , B , and C do not all lie in the same straight line, we may also make use of our sense of vision to determine when a fourth particle D lies in the same plane as A , B , and C . The test is as follows:

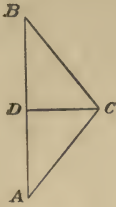
We take a fifth particle E , and then, if D lies in the plane of A , B , C , we may place E so that it is in the same straight line as D and one of the three particles, say A , and is also in the same straight line as the remaining particles B and C . (In case AD and BC are parallel, we must interchange either A and B or A and C in order to carry out the test.)

Simple Optical interpretations of like nature may be devised for various other Geometrical conceptions. As regards the notions and axioms of congruence these may be given a very simple interpretation by means of the properties of plane mirrors, but from a theoretical standpoint it appears better to regard congruence as based on the finite propagation of light.

We shall regard it as an experimental fact that light takes a finite interval of time in travelling from a particle A to a particle B and back again.

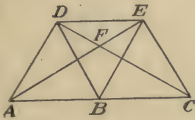
Let us suppose that we have a particle A which is so situated with respect to other particles B , C , D , &c., that a flash of light being sent out simultaneously to B , C , D , &c., and reflected from these back to A , returns to the latter simultaneously from all the particles. We shall then define the stretches AB , AC , AD , &c., as *congruent* or *equal*.

We shall define a right angle as follows :



Let D be a particle which lies in the same straight line as two others A and B , and so that the stretches DA and DB are equal. Let C be another particle not lying in the line AB , but such that the stretches CA and CB are equal. Then we shall define the angles CDA and CDB as right angles. As regards other angles we shall define them as congruent when their trigonometrical ratios are equal. The method of determining these will be obvious hereafter.

Since we have ascribed a meaning to the equality of stretches, we know the meaning of an equilateral triangle.



Suppose now we have five particles A, B, C, D, E , of which A, B , and C lie in a straight line and so that the stretches BA and BC are equal. Let the other two particles be so situated that ADB and BEC are equilateral triangles, and, further, let them lie in one plane and on the same side of the line AB .

The test for this is that we should be able to place a sixth particle F so as to lie both in the line AE and in the line CD .

We shall suppose then that observation shows our system to be such that DBE is also an equilateral triangle.

This excludes the Geometries of Lobatschewskij and Riemann, since in these the angle of an equilateral triangle is either less or more than one-third of two right angles. We shall find an interpretation of Lobatschewskij's Geometry when we come to deal with motion.

It will be shown that a system of three particles diverging uniformly, with equal relative rapidities, from simultaneous contact may be regarded as the corners of a Lobatschewskij triangle.

It thus appears that the fulfilment of our criterion of a Euclidian system excludes the possibility of our system of particles diverging from one another in this way—a possibility which at first sight might appear to lie open.

Since we are not yet in a position to prove this, we must ignore the seeming possibility until we have shown that the set of diverging particles has this property, and it may then be shown (by a process of *reductio ad absurdum*) that the seeming possibility has already actually been excluded.

There is another important restriction which we must suppose placed on our system of permanent configuration before we can go on to consider the motion of particles.

Let us consider any three particles A , B , C of our system, which are not all in the same straight line, and suppose a flash of light, starting out from one of the particles, say A , goes thence to B , thence to C , and thence back to A . Imagine a second flash, starting simultaneously with the first, and going round in the opposite direction, namely, from A to C , from C to B , and from B back to A .

Now, from the *purely logical standpoint*, three possibilities are open :

- (1) The first flash may arrive before the second.
- (2) The second flash may arrive before the first.
- (3) The two flashes may arrive simultaneously.

If, starting from any of the three particles, both flashes return simultaneously, we regard the system as not rotating in its own plane.

Thus "*absolute rotation*" acquires a definite meaning in our system of Optical Geometry. We shall suppose this test to be applied to the various sets of three particles which may be selected from a group of four particles which do not all lie in one plane. By doing this we ensure the absence of rotation about *any* axis of our system of permanent configuration.

Let us now suppose that any selected two of the particles of the system are at one-half the unit distance apart. As we have not yet properly defined distance, it would perhaps be better to say that, having selected a suitable pair of particles of the system, we define them to be at one-half the unit distance apart. Having got such a system of particles, we now proceed to make use of it.

"INDEX" OF FUNDAMENTAL PARTICLE AT ANY INSTANT.

We propose to introduce a conception which we shall call *the index of a particle at any instant*. This is a number which we shall define in a certain physical way as associated with a particle at an instant. We shall at first define it only for one of our fundamental particles, leaving the general definition till later.

Let us take then the particle A , and at a selected instant suppose a flash of light sent out to B , which is at one-half the unit distance from it, and reflected from B back to A . We shall say that the *index* of A at the instant of departure is 0, while at the instant of return it is 1. We shall suppose the flash returned once more at B and then back to A ,

and shall say that the index of A is then 2. We shall define the index of A at the n^{th} return as n . We may obviously, in a similar way, have negative values of the index.

We may interpolate indices to any desired extent by employing other particles instead of B which are "nearer" to A than is the latter. Thus, if we employed a particle B' such that light went to and returned from it to A ten times during the interval that light went to and returned from B to A once, we could assign indices to A differing by $\cdot 1$. Similarly it is theoretically possible to carry this process out indefinitely and ultimately to treat the index of A as a continuum in the usual manner.

It will be seen that the index may be regarded as a measure of time *so long as we confine our attention to the neighbourhood of the one particle A* , but it appears desirable in view of further developments not to identify index and time, as the two conceptions seem to run to some extent counter to one another.

DEFINITION OF THE INDEX FOR PARTICLES IN GENERAL.

Suppose a flash of light to go out from the particle A to some other particle P , which may be anywhere in space and may be "in motion."

Suppose the index of A at the instant of departure be denoted by N_a and called the *index of departure*.

Let the flash be reflected back from P to A and let the index of A at the instant of return be denoted by N_r and called the *index of return*.

We shall speak of the *index of arrival* of the flash at P , meaning thereby the index of the particle P at the instant of arrival of the light, and shall denote it by N_a .

The index of arrival is defined by the following equation:

$$N_a = \frac{d(N_r N_a)}{d(N_r + N_a)} \cdot \sqrt{\left(2 \frac{d^2(N_r N_a)}{d(N_r + N_a)^2}\right)}$$

If we denote $\frac{N_r N_a}{2}$ by u and $\frac{N_r + N_a}{2}$ by t , we may express N_a more simply thus

$$N_a = \frac{\frac{du}{dt}}{\sqrt{\left(\frac{d^2 u}{dt^2}\right)}}$$

It will be found that the index, as thus defined, possesses remarkable properties.

We shall consider first a system which is permanent in configuration, and not rotating.

We shall define the *distance* of any particle P from A as measured by

$$\frac{N_r - N_d}{2}.$$

In our permanent system this quantity remains fixed for each particle. If then we select any particle P , and let

$$\frac{N_r - N_d}{2} = k,$$

we have
$$N_r N_d = \left(\frac{N_r + N_d}{2} \right)^2 - k^2.$$

Thus
$$\frac{d(N_r N_d)}{d(N_r + N_d)} = \frac{N_r + N_d}{2},$$

and
$$2 \frac{d^2(N_r N_d)}{d(N_r + N_d)^2} = 1.$$

This gives

$$\frac{\frac{d(N_r N_d)}{d(N_r + N_d)}}{\sqrt{\left(2 \frac{d^2(N_r N_d)}{d(N_r + N_d)^2} \right)}} = \frac{N_r + N_d}{2} = N_a.$$

Thus the index of arrival in a non-rotating system of permanent configuration is the arithmetic mean of the indices of departure and return for light coming from the fundamental particle.

It is easy to see that this holds in general for such a system, and that any particle of it may take the place of the fundamental particle.

In dealing, however, with particles which are in relative motion, we must fix our attention on one particle and regard it throughout our investigations as the fundamental particle.

In particular, in the case of a rotating system, since light takes different intervals in going round a system of three particles in opposite directions, we should not get a unique value for the index at a particular instant if we were to vary our fundamental particle.

It is to be observed that the above result does not necessarily imply that the *instant* of arrival is the same as the instant at which the particle *A* has the index $\frac{N_r + N_d}{2}$. In fact, our definition of *index* is quite compatible with the light taking a different time in going from that which it takes to return.*

UNIFORM MOTION OF PARTICLES.

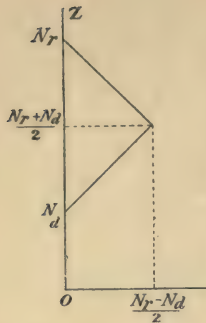
In the consideration of particles in motion, we refer them to systems of permanent configuration which are non-rotating.

If we send out a flash of light from some particle *A* in such a system to a moving particle *P*, the light will reach the latter when it has a definite position in the system. If we imagine a particle *B* of the system to occupy as nearly as possible this position, then *B* is at a definite distance from the particle *A*, and this distance is $\frac{N_r - N_d}{2}$. Also at the instant when the light reaches *P*, which is also the instant at which its position coincides with that of *B*, the particle *B* has the index $\frac{N_r + N_d}{2}$.

Thus for each value of the index in the system of permanent configuration, the particle *P* occupies some definite position in the latter. The index of *P* is, however, quite different from that of *B* at the instant when they coincide.

It will be found of great assistance, when the motion of the particle is confined to one plane, to take the two co-ordinates *x* and *y* as representing the position of the moving particle with respect to our system of permanent configuration; while the third co-ordinate *z* represents the index of particles belonging to the system. The aggregate of positions of a moving particle will be represented by some continuous line; while the light going from one particle to another is represented by a straight line making an angle of 45° with the axis of *z*.

* This point is philosophically very important, since the *theory of relativity*, as usually presented, involves the psychological difficulty of a "time" which is not unique. Whether or no there may be some deep sense in which this is true, I am not prepared to say, nevertheless this difficulty is very likely to stand in the way of a general acceptance of the theory. It rather seems to the writer that the assumption of a unique time is intimately bound up with the logical *principle of non-contradiction*, whereby a thing cannot both be and not be at the same time. The conception of the index of a particle at an instant appears to avoid this difficulty.



It is easily seen that this representation fits in with the result already obtained that the index of arrival in a system of permanent configuration which is not rotating is equal to the arithmetic mean of the indices of departure and return. It also fits in with our definition of the measure of the distance between two particles in such a system as $\frac{N_r - N_d}{2}$. This is clear from the diagram.

It is, however, to be noted that distance defined in this way is absolute distance, and is always positive.

It is easily seen that all the straight lines through a point (a, b, c) which make angles of 45° with the axis of z lie on the cone

$$(x - a)^2 + (y - b)^2 - (z - c)^2 = 0.$$

We shall refer to this as the standard cone with respect to the point (a, b, c) .

If a particle P be in motion with respect to the system of permanent configuration, there is always a tangent to its path at any instant. If C be any particle of the system from which the direction of P appears stationary at any instant, and if a flash of light goes out from C to P and back, the absolute velocity of P with respect to C is defined to be $\left| \frac{d(N_r - N_d)}{d(N_r + N_d)} \right|$ at the instant of arrival of the light. This may also be called the absolute velocity of P with respect to the system.

A particle in uniform motion in a straight line with respect to the system will be represented by a straight line inclined to the axis of z at some angle whose tangent will represent the absolute velocity with respect to that system. This angle we shall suppose to be less than 45° , since, if it were greater than this, a flash of light sent out from C would never reach the particle P if the instant of departure of the light were later than the instant at which P leaves the position of C .

RAPIDITY OF A PARTICLE WITH RESPECT TO A SYSTEM.

We propose now to define what we shall call the *rapidity* of a particle with respect to a system of permanent configuration.

If v be the absolute velocity of the particle with respect to the system, then the inverse hyperbolic tangent of v will be spoken of as the rapidity.

Thus if ω be the rapidity,

$$v = \tanh \omega.$$

As ω increases from 0 to ∞ , v increases from 0 to 1.

For small values of ω we have, practically, velocity is equal to rapidity, but we shall see later that, for large values, it is the rapidity and not the velocity which follows the additive law.

INDEX OF A PARTICLE MOVING WITH UNIFORM VELOCITY IN A STRAIGHT LINE.

We shall consider a plane containing the line of motion of the particle and some particle of the system of constant configuration, and with this particle as origin shall take axes of x and y . If now we take a three-dimensional set of rectangular axes Ox , Oy , Oz , and let the z co-ordinate represent the index of the particles in the system of constant configuration, the moving particle will be represented by a line whose equations may be taken as

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r,$$

where (x_1, y_1, z_1) are the co-ordinates of some point of the line and l, m, n are its direction cosines.

The standard cone, with respect to any point on this line, is

$$(x - lr - x_1)^2 + (y - mr - y_1)^2 - (z - nr - z_1)^2 = 0.$$

This intersects the axis of z in two points, whose z co-ordinates are the roots of the equation

$$z^2 - 2(nr + z_1)z + (nr + z_1)^2 - (lr + x_1)^2 - (mr + y_1)^2 = 0.$$

Denoting the smaller of these roots by N_d and the larger by N_r , we have

$$N_r N_d = (nr + z_1)^2 - (lr + x_1)^2 - (mr + y_1)^2$$

and
$$N_r + N_d = 2(nr + z_1).$$

Suppose now a small increase of r to take place, we have

$$\begin{aligned} d(N_r N_d) &= 2 \{ (nr + z_1)n - (lr + x_1)l - (mr + y_1)m \} dr \\ &= 2 \{ (2n^2 - 1)r + z_1 n - x_1 l - y_1 m \} dr \end{aligned}$$

and
$$d(N_r + N_a) = 2n dr.$$

Thus
$$\frac{d(N_r N_a)}{d(N_r + N_a)} = \frac{2n^2 - 1}{n} r + z_1 - \frac{x_1 l + y_1 m}{n}.$$

If now we put for r its value $\frac{z - z_1}{n}$, where z is the value of that co-ordinate for the vertex of the cone, we get

$$\frac{d(N_r N_a)}{d(N_r + N_a)} = \left(2 - \frac{1}{n^2}\right) z + \frac{1 - n^2}{n^2} z_1 - \frac{x_1 l + y_1 m}{n}.$$

Now it is easy to show that if \bar{z} be the z co-ordinate of the point of the line which is nearest to the axis of z then

$$\bar{z} = \frac{(1 - n^2) z_1 - n(x_1 l + y_1 m)}{1 - n^2}.$$

Thus
$$\frac{d(N_r N_a)}{d(N_r + N_a)} = \left(2 - \frac{1}{n^2}\right) z + \frac{1 - n^2}{n^2} \bar{z}$$

or (if $n = \cos \gamma$)
$$= z - \tan^2 \gamma (z - \bar{z}).$$

Also, since $\frac{N_r + N_a}{2} = z$, we have

$$2 \frac{d^2(N_r N_a)}{d(N_r + N_a)^2} = 1 - \tan^2 \gamma,$$

and accordingly

$$N_a \equiv \frac{\frac{d(N_r N_r)}{d(N_r + N_a)}}{\sqrt{\left(2 \frac{d^2(N_r N_a)}{d(N_r + N_a)^2}\right)}} = \frac{z - \tan^2 \gamma (z - \bar{z})}{\sqrt{(1 - \tan^2 \gamma)}}.$$

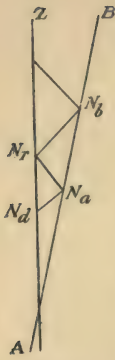
If we write $(z - \bar{z}) \tan \gamma = s$, and put $\tan \gamma = \tanh \omega$, where ω is the rapidity of the moving particle with respect to the system of permanent configuration, we have

$$\sqrt{\left(2 \frac{d^2(N_r N_a)}{d(N_r + N_a)^2}\right)} = \sqrt{(1 - \tanh^2 \omega)} = \frac{1}{\cosh \omega}.$$

Thus
$$N_a = z \cosh \omega - s \sinh \omega.$$

If we imagine a line drawn parallel to the axis of z through the point of the line representing the moving particle at the point where it is nearest to the axis of z , then it is easy

RECIPROCAL RELATION.



We have seen that in a system of permanent configuration we may take any particle as our fundamental particle, and the indices defined with respect to that particle may be employed to work back from any other particle to the fundamental one.

In defining the index of the fundamental particle, however, we selected a perfectly arbitrary instant for the zero of index. Thus, as in the case of *potential*, it is *difference of potential* which is of physical importance; so in the case of *index* it is *difference of index*.

We propose now to show that if we have a particle which moves with constant velocity with respect to our system of permanent configuration, and we take the index of the particle with respect to our fundamental particle, we may use the index so obtained to work back to the index of the fundamental particle, if the zero of index has been properly chosen.

If a different zero has been chosen we obtain, not the index of the fundamental particle itself, but the index plus a constant. This constant, however, cancels out if we are dealing with *differences of index*. In order to show this, we again make use of our geometrical representation.

Consider then the standard cone taken with respect to a point on the z axis, say N_r . Its equation is

$$x^2 + y^2 - (z - N_r)^2 = 0.$$

This cone meets the line AB representing the moving particle in two points, the z 's of which are given by the equation

$$\left\{ \frac{l}{n} (z - z_1) + x_1 \right\}^2 + \left\{ \frac{m}{n} (z - z_1) + y_1 \right\}^2 - \{z - N_r\}^2 = 0,$$

or

$$\frac{2n^2 - 1}{n^2} z^2 - 2 \left\{ N_r + \frac{lx_1 + my_1}{n} - \frac{l + m}{n} z_1 \right\} z$$

$$+ N_r^2 - x_1^2 - y_1^2 + 2 \frac{lx_1 + my_1}{n} z_1 - \frac{1 - n^2}{n^2} z_1^2 = 0.$$

Multiplying through by $\frac{n^2}{2n^2 - 1}$ we may write this equation in the form

$$z^2 - 2 \frac{n^2}{2n^2 - 1} (N_r + A) + \frac{n^2}{2n^2 - 1} (N_r^2 + B) = 0,$$

where A and B are constants for the line AB .

Now let z' and z'' be the roots of this equation. We have

$$z' + z'' = \frac{2n^2}{2n^2 - 1} (N_r + A)$$

and

$$z' z'' = \frac{n^2}{2n^2 - 1} (N_r^2 + B).$$

If now N_a and N_b be the indices of the moving particle corresponding to these two points, we have, in accordance with the result obtained in the previous section,

$$N_a = \frac{z' - \tan^2 \gamma (z' - \bar{z})}{\sqrt{(1 - \tan^2 \gamma)}} \\ = \frac{\frac{2n^2 - 1}{n^2} z' + \frac{1 - n^2}{n^2} \bar{z}}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}}$$

and

$$N_b = \frac{\frac{2n^2 - 1}{n^2} z'' + \frac{1 - n^2}{n^2} \bar{z}}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}}.$$

Thus

$$N_a N_b = \frac{2n^2 - 1}{n^2} z' z'' + \frac{1 - n^2}{n^2} \bar{z} (z' + z'') + \frac{(1 - n^2)^2}{n^2 (2n^2 - 1)} \bar{z}^2 \\ = N_r^2 + B + 2 \frac{1 - n^2}{2n^2 - 1} \bar{z} (N_r + A) + \frac{(1 - n^2)^2}{n^2 (2n^2 - 1)} \bar{z}^2$$

and

$$N_a + N_b = \frac{\frac{2n^2 - 1}{n^2} (z' + z'') + \frac{2(1 - n^2)}{n^2} \bar{z}}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}} \\ = \frac{2(N_r + A) + \frac{2(1 - n^2)}{n^2} \bar{z}}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}}.$$

Suppose now a small increase to take place in N_r and we get

$$d(N_a N_b) = \left\{ N_r + \frac{1 - n^2}{2n^2 - 1} \bar{z} \right\} 2dN_r$$

and
$$d(N_a + N_b) = \frac{2dN_r}{\sqrt{\left(\frac{2n^2-1}{n^2}\right)}}$$

Thus
$$\frac{d(N_a N_b)}{d(N_a + N_b)} = \frac{N_r + \frac{1-n^2}{2n^2-1} \bar{z}}{\sqrt{\left(\frac{n^2}{2n^2-1}\right)}}$$

and
$$2 \frac{d^2(N_a N_b)}{d(N_a + N_b)^2} = \frac{2n^2-1}{n^2} = \frac{1}{\cosh^2 \omega}$$

Thus
$$\frac{\frac{d(N_a N_b)}{d(N_a + N_b)}}{\sqrt{\left(2 \frac{d^2(N_a N_b)}{d(N_a + N_b)^2}\right)}} = N_r + \frac{1-n^2}{2n^2-1} \bar{z}$$

$$= N_r + \sinh^2 \omega \bar{z}.$$

Thus we see that if the zero of index of the fundamental particle be chosen so that $\bar{z}=0$, we get

$$\frac{\frac{d(N_a N_b)}{d(N_a + N_b)}}{\sqrt{\left(2 \frac{d^2(N_a N_b)}{d(N_a + N_b)^2}\right)}} = N_r.$$

Thus, if the zero of index be that for which the two particles are *nearest* to one another, we may work back from the moving particle to the fundamental one, and the indices are connected by a reciprocal relation.

In particular, if the two particles be such that at any instant they are in contact, we may select that instant as the one at which both particles have the index zero.

ARITHMETIC MEAN THEOREM.

We propose now to show that if we have a second particle moving in the same direction as the first and having the same velocity, and if we send out a flash of light from the one particle to the other and back again to the first, then the index of arrival is equal to the arithmetic mean of the indices of departure and return.

Let us take our fundamental particle as lying in the plane of the two moving particles and we shall then be able,

as before, to represent index in our system of permanent configuration by a z axis perpendicular to those of x and y , which give the positions in the plane.

Let the one moving particle be represented by the line

$$\frac{x - \bar{x}_1}{l} = \frac{y - \bar{y}_1}{m} = \frac{z - \bar{z}_1}{n}$$

and the other by

$$\frac{x - \bar{x}_2}{l} = \frac{y - \bar{y}_2}{m} = \frac{z - \bar{z}_2}{n},$$

where $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$ are the co-ordinates of the points which are nearest to the axis of z .

Take a point of the first line defined by r_1 , so that its co-ordinates are

$$lr_1 + \bar{x}_1, \quad mr_1 + \bar{y}_1, \quad nr_1 + \bar{z}_1.$$

The standard cone with respect to this point is

$$(x - lr_1 - \bar{x}_1)^2 + (y - mr_1 - \bar{y}_1)^2 - (z - nr_1 - \bar{z}_1)^2 = 0.$$

Taking the points of intersection of this cone with the second line, we have

$$x = \frac{l(z - \bar{z}_2)}{n} + \bar{x}_2, \quad y = \frac{m(z - \bar{z}_2)}{n} + \bar{y}_2.$$

Thus, for the points of intersection we have

$$\begin{aligned} \{z - nr_1 - \bar{z}_1\}^2 - \left\{ \frac{l(z - \bar{z}_2)}{n} + \bar{x}_2 - \bar{x}_1 - lr_1 \right\}^2 \\ - \left\{ \frac{m(z - \bar{z}_2)}{n} + \bar{y}_2 - \bar{y}_1 - mr_1 \right\}^2 = 0. \end{aligned}$$

Expanding, we get

$$\begin{aligned} \frac{2n^2 - 1}{n^2} z^2 + 2 \frac{1 - n^2}{n^2} z \bar{z}_2 \\ - 2 \left\{ \frac{l}{n} (\bar{x}_2 - \bar{x}_1 - lr_1) + \frac{m}{n} (\bar{y}_2 - \bar{y}_1 - mr_1) + \bar{z}_1 + nr_1 \right\} z \\ + \{nr_1 - \bar{z}_1\}^2 - \left\{ \bar{x}_2 - \bar{x}_1 - lr_1 - \frac{l}{n} \bar{z}_2 \right\}^2 \\ - \left\{ \bar{y}_2 - \bar{y}_1 - mr_1 - \frac{m}{n} \bar{z}_2 \right\}^2 = 0. \end{aligned}$$

But it is easy to show that

$$l\bar{x}_2 + m\bar{y}_2 = 0$$

and

$$l\bar{x}_1 + m\bar{y}_1 = 0.$$

Thus the above may be written

$$\begin{aligned} & \left\{ \frac{2n^2-1}{n^2} z^2 + 2 \frac{1-n^2}{n} z\bar{z}_2 + \frac{(1-n^2)^2}{n^2(2n^2-1)} \bar{z}_2^2 \right\} - \frac{(1-n^2)^2}{n^2(2n^2-1)} \bar{z}_2^2 \\ & - 2 \left\{ \bar{z}_1 + \frac{2n^2-1}{n} r_1 \right\} \left\{ z + \frac{1-n^2}{2n^2-1} \bar{z}_2 \right\} \\ & + 2 \frac{1-n^2}{2n^2-1} \bar{z}_2 \left\{ \bar{z}_1 + \frac{2n^2-1}{n} r_1 \right\} \\ & + \{nr_1 - z_1\}^2 - \left\{ \bar{x}_2 - \bar{x}_1 - lr_1 - \frac{l}{n} \bar{z}_2 \right\}^2 \\ & - \left\{ \bar{y}_2 - \bar{y}_1 - mr_1 - \frac{m}{n} \bar{z}_2 \right\}^2 = 0, \end{aligned}$$

$$\begin{aligned} \text{or } & \left\{ \frac{2n^2-1}{n^2} z + \frac{1-n^2}{n^2} \bar{z}_2 \right\}^2 \\ & - \frac{2 \left\{ \bar{z}_1 + \frac{2n^2-1}{n} r_1 \right\} \left\{ \frac{2n^2-1}{n^2} z + \frac{1-n^2}{n^2} \bar{z}_2 \right\}}{\sqrt{\left(\frac{2n^2-1}{n^2}\right)} \sqrt{\left(\frac{2n^2-1}{n^2}\right)}} \\ & + \{nr_1 - \bar{z}_1\}^2 - \left\{ \bar{x}_2 - \bar{x}_1 - lr_1 - \frac{l}{n} \bar{z}_2 \right\}^2 - \left\{ \bar{y}_2 - \bar{y}_1 - mr_1 - \frac{m}{n} \bar{z}_2 \right\}^2 \\ & + \frac{2(1-n^2)}{2n^2-1} \bar{z}_2 \left\{ \bar{z}_1 + \frac{2n^2-1}{n} r_1 \right\} - \frac{(1-n^2)^2}{n^2(2n^2-1)} \bar{z}_2^2 = 0. \end{aligned}$$

But this is a quadratic in

$$\frac{\frac{2n^2-1}{n^2} z + \frac{1-n^2}{n^2} \bar{z}_2}{\sqrt{\left(\frac{2n^2-1}{n^2}\right)}}$$

and the two values of this are the indices of the second moving particle at the instants of departure and return of light going from it to the first moving particle. If we call the two roots of this equation N_d and N_r , we have

$$\frac{N_d + N_r}{2} = \frac{\frac{2n^2 - 1}{n} r_1 + \bar{z}_1}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}}.$$

But if (x_1, y_1, z_1) be the point on the first line corresponding to r_1 , we have

$$\frac{z - \bar{z}_1}{n} = r_1.$$

Thus

$$\frac{N_d + N_r}{2} = \frac{\frac{2n^2 - 1}{n^2} z_1 + \frac{1 - n^2}{n^2} \bar{z}_1}{\sqrt{\left(\frac{2n^2 - 1}{n^2}\right)}}.$$

The expression on the right is however the index of the first particle at the instant of arrival of the light from the second particle. Calling this N_a , we have

$$\frac{N_d + N_r}{2} = N_a,$$

or the index of arrival is equal to the arithmetic mean of the indices of departure and return.

This is the same as the result which we obtained for the case of two particles in our original system of permanent configuration, and we now find that it also holds for the case of two particles moving in the same direction and with the same velocity with respect to that system.

If the fundamental particle does not lie in the plane containing the lines of motion of the particles, we can no longer represent index by the third dimension, but the demonstration is quite analogous.

Instead of the standard cone

$$(x - a)^2 + (y - b)^2 - (z - c)^2 = 0,$$

we take $(x - a)^2 + (y - b)^2 + (z - c)^2 - (w - d)^2 = 0$,

where w now represents index in the system of permanent configuration. The particles then satisfy the equations

$$\frac{x - \bar{x}_1}{l} = \frac{y - \bar{y}_1}{m} = \frac{z - \bar{z}_1}{n} = \frac{w - \bar{w}_1}{k},$$

and

$$\frac{x - \bar{x}_2}{l} = \frac{y - \bar{y}_2}{m} = \frac{z - \bar{z}_2}{n} = \frac{w - \bar{w}_2}{k},$$

where $l^2 + m^2 + n^2 + k^2 = 1$.

The demonstration then proceeds exactly as before, except that we have now three co-ordinates representing ordinary space instead of two.

ABSENCE OF ROTATION IN UNIFORMLY MOVING SYSTEM.

In obtaining our original system of constant configuration we made use of a test for absence of rotation, by sending a flash of light in opposite directions round a set of three particles.

We now propose to show that if we have a set of three particles which move in the same direction with respect to our original system, with equal velocities, then the same test will be satisfied.

We may, as before, represent index in the original system by means of the z co-ordinate and shall suppose the motion to be in the x direction.

A particle moving in this direction will be represented by a line parallel to the plane of x, z .

The equations of such a line may be written

$$x - x_0 = z \tan \gamma,$$

$$y = y_0,$$

where x_0, y_0 are the x and y co-ordinates of the point where the line cuts the plane of x, y .

Consider now the intersection of the cone

$$x^2 + y^2 - z^2 = 0$$

with this line.

We have

$$z^2 - (z \tan \gamma + x_0)^2 - y_0^2 = 0$$

or

$$(1 - \tan^2 \gamma) z^2 - 2x_0 \tan \gamma z - x_0^2 - y_0^2 = 0.$$

The positive root of this equation in z is

$$z = \frac{x_0 \tan \gamma + \sqrt{\{x_0^2 + y_0^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma}.$$

Consider now the three parallel lines meeting the plane of x, y in the points

$$(0, 0), (x', y'), (x'', y''),$$

and call these lines $A, B,$ and C respectively.

Starting from the origin and going round in the order $A - B - C - A$ we get, adding the successive increases of $z,$

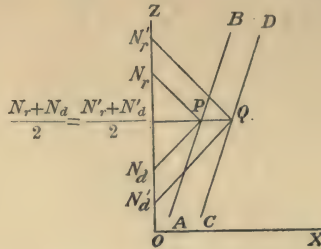
$$\begin{aligned} & \frac{x' \tan \gamma + \sqrt{\{x'^2 + y'^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & + \frac{(x'' - x') \tan \gamma + \sqrt{\{(x'' - x')^2 + (y'' - y')^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & + \frac{-x'' \tan \gamma + \sqrt{\{x''^2 + y''^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & = \frac{\left[\begin{array}{c} \sqrt{\{x'^2 + y'^2 (1 - \tan^2 \gamma)\}} \\ + \sqrt{\{(x'' - x')^2 + (y'' - y')^2 (1 - \tan^2 \gamma)\}} + \sqrt{\{x''^2 + y''^2 (1 - \tan^2 \gamma)\}} \end{array} \right]}{1 - \tan^2 \gamma}. \end{aligned}$$

Next, starting from the origin and going round in the order $A - C - B - A$ we get, as before,

$$\begin{aligned} & \frac{x'' \tan \gamma + \sqrt{\{x''^2 + y''^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & + \frac{-(x'' - x') \tan \gamma + \sqrt{\{(x'' - x')^2 + (y' - y'')^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & + \frac{-x' \tan \gamma + \sqrt{\{x'^2 + y'^2 (1 - \tan^2 \gamma)\}}}{1 - \tan^2 \gamma} \\ & = \frac{\left[\begin{array}{c} \sqrt{\{x''^2 + y''^2 (1 - \tan^2 \gamma)\}} \\ + \sqrt{\{(x'' - x')^2 + (y' - y'')^2 (1 - \tan^2 \gamma)\}} + \sqrt{\{x'^2 + y'^2 (1 - \tan^2 \gamma)\}} \end{array} \right]}{1 - \tan^2 \gamma}. \end{aligned}$$

Thus the result is the same whichever way we go round, and so we see that flashes of light starting out simultaneously and going round in opposite directions would arrive simultaneously.

APPARENT CHANGE OF DIMENSIONS DUE TO MOTION.



If we have two particles moving with the same velocity and in the same direction with respect to a system of permanent configuration we may represent them by parallel straight lines.

In order to get our ideas clear we shall suppose the two particles to move in the direction of the x axis and take the z axis to represent indices of our fundamental particle.

Let AB and CD represent the two moving particles and suppose them to pass through the positions P and Q respectively, where, if there were particles of the permanent system, such particles would have equal indices. The distance between two such particles of the permanent system is the *apparent distance* between the two moving particles as viewed from the permanent system. This is, however, different from the distance between the two moving particles as defined by the quantity $\frac{N_r - N_d}{2}$ between the two.

The result of this is that if we have a system of particles which are all moving with the same uniform velocity and in the same direction with respect to a system of permanent configuration then a sphere in the moving system is apparently an oblate spheroid as viewed from the former.

Since the effect is symmetrical about the direction of motion, we may, as before, consider merely the plane of x, y and make use of the axis of z to represent index in our system of permanent configuration.

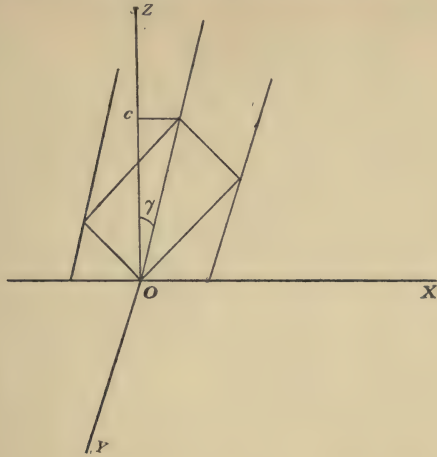
Consider a line in the plane of x, z , whose equations are

$$x = z \tan \gamma,$$

$$y = 0,$$

and let this represent a particle moving with a velocity equal to $\tan \gamma$ with respect to our permanent system. Suppose now a flash of light to go out from this particle

to a ring of particles moving with it in the plane of x, y , and suppose the ring to be of such a form that the light returns from all the particles simultaneously.



Consider the intersection of two standard cones, the one with respect to the point $(0, 0, 0)$ and the other with respect to the point $(c \tan \gamma, 0, c)$.

The first cone is

$$x^2 + y^2 - z^2 = 0.$$

The other is

$$(x - c \tan \gamma)^2 + y^2 - (z - c)^2 = 0.$$

The common points lie on the surface

$$x^2 + y^2 - z^2 - \{(x - c \tan \gamma)^2 + y^2 - (z - c)^2\} = 0$$

or

$$z - \tan \gamma x - \frac{c}{2}(1 - \tan^2 \gamma) = 0.$$

This is the equation of a plane making an angle γ with the axis of x .

Again the common points of the two cones must lie on the surface

$$x^2 + y^2 - z^2 + \frac{1}{1 - \tan^2 \gamma} \left\{ z - \tan \gamma x - \frac{c}{2}(1 - \tan^2 \gamma) \right\}^2 + c \left\{ z - \tan \gamma x - \frac{c}{2}(1 - \tan^2 \gamma) \right\} = 0$$

or

$$\frac{x^2}{1 - \tan^2 \gamma} + y^2 + \frac{\tan^2 \gamma z^2}{1 - \tan^2 \gamma} - \frac{2 \tan \gamma}{1 - \tan^2 \gamma} xz - \frac{c^2}{4} (1 - \tan^2 \gamma) = 0.$$

This may be written in the form

$$\frac{(x - \tan \gamma z)^2}{\left\{ \frac{1}{2} c (1 - \tan^2 \gamma) \right\}^2} + \frac{y^2}{\frac{1}{4} c^2 (1 - \tan^2 \gamma)} = 1.$$

This is the equation of an elliptic cylinder whose generators are parallel to the original line, and which therefore represent particles moving in the same direction as the original moving particle and with the same velocity.

The light going out simultaneously from the latter to all particles represented by generators of this cylinder will return simultaneously to it.

If we put $z = 0$ we get the apparent form of this ring of particles as viewed from the permanent system. We get

$$\frac{x^2}{\left\{ \frac{1}{2} c (1 - \tan^2 \gamma) \right\}^2} + \frac{y^2}{\frac{1}{4} c^2 (1 - \tan^2 \gamma)} = 1.$$

This is an ellipse in the plane of x, y , the ratio of whose axes is $\sqrt{(1 - \tan^2 \gamma)}:1$. If we put $c = \frac{2}{\sqrt{(1 - \tan^2 \gamma)}}$, this becomes

$$\frac{x^2}{1 - \tan^2 \gamma} + y^2 = 1.$$

But the index of the original moving particle corresponding to the point with respect to which the second standard cone was taken is

$$c \sqrt{(1 - \tan^2 \gamma)} = 2.$$

Also the index of this particle corresponding to the instant of departure of the light is zero.

Thus for the moving system we have

$$\frac{N_r - N_d}{2} = 1.$$

The demonstration for the case where the particles are not confined to one plane is quite analogous.

APPARENT CHANGE OF ANGLES BY MOTION.

From the above it appears that lengths, as measured by $\frac{N_r - N_d}{2}$, suffer no apparent alteration in a direction at right angles to that of motion, but, as observed from the original system, they appear to be shortened in the direction of motion.

This involves an apparent change of angle as determined from the trigonometrical ratios. Thus if θ' be the angle between the direction of relative motion and a direction fixed with respect to the moving particle while θ is the corresponding angle observed from the moving particle we have

$$\tan \theta' = \frac{p}{q'}, \text{ say,}$$

and

$$\tan \theta = \frac{p}{q},$$

where

$$\frac{q'}{q} = \sqrt{1 - \tan^2 \gamma} = \frac{1}{\cosh \omega},$$

where ω is the rapidity. Thus

$$\tan \theta = \frac{\tan \theta'}{\cosh \omega},$$

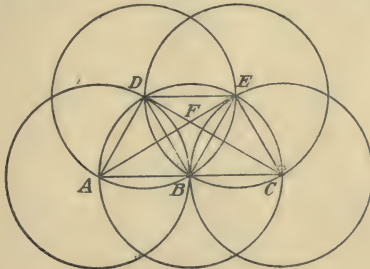
THE GEOMETRY OF A UNIFORMLY MOVING SYSTEM OF PERMANENT CONFIGURATION IS EUCLIDIAN.

A test has already been described to show that the Geometry of our original system of permanent configuration was Euclidian.

We suppose three particles A , B , and C to be taken in a straight line and such that A and C were equidistant from B . Two other particles D and E were then supposed taken so that ADB and BEC were equilateral triangles and D and E both in one plane with the particles A , B , C and on the same side of the line AB .

We supposed then that observation showed the triangle BDE was also equilateral.

The test that the particles should lie in one plane was that it should be possible to place a sixth particle F so as to be in the same straight line as A and E and also in the same straight line as C and D . We may imagine a number of circles as shown in the figure with centres A , B , C , D ,



and E and radii equal to a side of one of the equilateral triangles. We may suppose the whole figure to be projected orthogonally upon a plane inclined to its own at an angle whose cosine is $\sqrt{1 - \tan^2 \gamma}$, and then all the circles are projected into similar equal and similarly situated ellipses, the ratio of whose axes will all be as $1 : \sqrt{1 - \tan^2 \gamma}$. If the plane of these ellipses be taken as that of x, y , and if A', B', C', D', E', F' be the projections of A, B, C, D, E, F respectively, and if straight lines be taken through $A', B', C', \&c.$, which are perpendicular to the line of intersection of the two planes, and all make angles γ with the axes of z , then these lines will represent particles which are all in motion in the same direction with a velocity equal to $\tan \gamma$. The lines through A', B' , and C' lie in one plane, and so the particles which these lines represent will all lie in the same straight line. Similarly, for the particles represented by the lines through A', F' , and E' and also for those represented by the lines through C', F', D' . Further, since the ellipses in the plane of x, y all show the contraction due to a uniform velocity $\tan \gamma$, the three triangles in the moving system, whose corners are represented by the lines through A', D', B' , those through B', E', C' , and those through B', D', E' , have their sides all equal. Thus the Geometry of the moving system is Euclidian.

COMPOSITION OF RAPIDITIES.

We have already seen that if we have a particle which moves with uniform velocity in a straight line with respect to the fundamental particle, in such a way that the two particles are in contact at a certain instant, then if we take that instant as that at which the index of the fundamental particle is zero the process by which the index of the moving particle is obtained is a reciprocal one. Let us now consider two particles which are both in contact with the fundamental particle at the same instant.

The three particles in general define a plane which we shall take as the plane of x, y , while we shall, as before, represent the index of the fundamental particle by the z co-ordinate.

Let the one particle be represented by the line

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$$

and the other by

$$\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$$

Take a point (x_2, y_2, z_2) on the second line and take the standard cone with respect to it.

We have

$$x_2 = \frac{l_2}{n_2} z_2$$

and

$$y_2 = \frac{m_2}{n_2} z_2,$$

Thus the cone is

$$(z - z_2)^2 - \left(x - \frac{l_2}{n_2} z_2\right)^2 - \left(y - \frac{m_2}{n_2} z_2\right)^2 = 0.$$

If (x_1, y_1, z_1) be a point where this cone meets the other line we have

$$(z_1 - z_2)^2 - \left(\frac{l_1}{n_1} z_1 - \frac{l_2}{n_2} z_2\right)^2 - \left(\frac{m_1}{n_1} z_1 - \frac{m_2}{n_2} z_2\right)^2 = 0$$

$$\text{or } \frac{2n_1^2 - 1}{n_1^2} z_1^2 - 2 \frac{n_1 n_2 - l_1 l_2 - m_1 m_2}{n_1 n_2} z_1 z_2 + \frac{2n_2^2 - 1}{n_2^2} z_2^2 = 0.$$

If N_1 and N_2 represent indices of the corresponding particles we have

$$\sqrt{\left(\frac{2n_1^2 - 1}{n_1^2}\right)} z_1 = N_1, \quad \sqrt{\left(\frac{2n_2^2 - 1}{n_2^2}\right)} z_2 = N_2.$$

$$\text{Thus } N_1^2 - 2 \frac{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}}{\sqrt{\left(\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}\right)}} N_1 N_2 + N_2^2 = 0.$$

The two values of N_1 given by this equation are the indices of departure and return of light going from particle 1 to particle 2.

Calling these N_d and N_r we have

$$N_r N_d = N_2^2$$

$$\text{and } N_r + N_d = 2 \frac{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}}{\sqrt{\left(\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}\right)}} N_2.$$

Thus

$$d(N_r N_d) = 2N_2 dN_2$$

$$\text{and } d(N_r + N_d) = 2 \frac{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}}{\sqrt{\left(\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}\right)}} dN_2.$$

$$\text{Thus } \frac{d(N_r N_d)}{d(N_r + N_d)} = \frac{\sqrt{\left(\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}\right)}}{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}} N_2$$

$$\text{and } 2 \frac{d^2(N_r N_d)}{d(N_r + N_d)^2} = \frac{\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}}{\left\{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}\right\}^2}.$$

$$\text{Thus } \frac{\frac{d(N_r N_d)}{d(N_r + N_d)}}{\sqrt{\left(2 \frac{d^2(N_r N_d)}{d(N_r + N_d)^2}\right)}} = N_2 = N_a.$$

Thus the index of arrival of light coming from particle (1) to particle (2) may be obtained by the same formula as that by which the index of either particle is obtained from the fundamental one.

It is evident that this holds for as many particles as we please, provided that they are all in contact at the same instant.

Now we have already seen (see p. 10) that

$$2 \frac{d^2(N_r N_d)}{d(N_r + N_d)^2} = \frac{1}{\cosh^2 \omega}$$

for the case of a particle moving with constant rapidity ω with respect to the fundamental particle.

If, then, we refer to the fundamental particle by the suffix 3 (having already assigned suffixes 1 and 2 to the moving particles) and taking the three particles as the corners of a triangle, let ω_1 and ω_2 represent respectively the rapidity of 2 with respect to 3 and of 1 with respect to (3); while at the same time we write

$$\frac{1}{\cosh^2 \omega_3} = \frac{\frac{2n_1^2 - 1}{n_1^2} \frac{2n_2^2 - 1}{n_2^2}}{\left\{1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}\right\}^2}.$$

We have
$$\frac{1}{\cosh^2 \omega_3} = \frac{1}{\cosh^2 \omega_1} \frac{1}{\cosh^2 \omega_2} \cdot \left\{ 1 - \frac{l_1 l_2 + m_1 m_2}{n_1 n_2} \right\}^2.$$

We must now find the value of the quantity

$$\frac{l_1 l_2 + m_1 m_2}{n_1 n_2}.$$

If λ_1, μ_1, ν_1 be the direction cosines of a plane through the first line and the axis of z , we have

$$l_1 \lambda_1 + m_1 \mu_1 = 0,$$

$$\nu_1 = 0,$$

$$\lambda_1^2 + \mu_1^2 = 1.$$

Thus

$$\lambda_1 = \pm \frac{m_1}{\sqrt{(l_1^2 + m_1^2)}}$$

$$= \pm \frac{m_1}{\sqrt{(1 - n_1^2)}},$$

and

$$\mu_1 = \mp \frac{l_1}{\sqrt{(1 - n_1^2)}}.$$

Similarly, if λ_2, μ_2, ν_2 be the direction cosines of a plane through the second line and the axis of z , we have

$$\lambda_2 = \pm \frac{m_2}{\sqrt{(1 - n_2^2)}},$$

$$\mu_2 = \mp \frac{l_2}{\sqrt{(1 - n_2^2)}},$$

$$\nu_2 = 0.$$

If then Ω_3 be the angle between these planes, we have

$$\cos \Omega_3 = \pm \frac{l_1 l_2 + m_1 m_2}{\sqrt{(1 - n_1^2)} \sqrt{(1 - n_2^2)}}.$$

Thus
$$\pm \sqrt{\left(\frac{1 - n_1^2}{n_1^2}\right)} \sqrt{\left(\frac{1 - n_2^2}{n_2^2}\right)} \cos \Omega_3 = \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}$$

or
$$\pm \tanh \omega_1 \tanh \omega_2 \cos \Omega_3 = \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}.$$

If we select the angle Ω_3 so that $\Omega_3 = 0$ corresponds to the same side of the axis of z we take the upper sign.

Thus $\tanh \omega_1 \tanh \omega_2 \cos \Omega_3 = \frac{l_1 l_2 + m_1 m_2}{n_1 n_2}$.

which gives

$$\frac{1}{\cosh^2 \omega_3} = \frac{1}{\cosh^2 \omega_1} \frac{1}{\cosh^2 \omega_2} \frac{1}{\{1 - \tanh \omega_1 \tanh \omega_2 \cos \Omega_3\}^2}.$$

Extracting square roots, we get finally

$$\cosh \omega_3 = \cosh \omega_1 \cosh \omega_2 - \sinh \omega_1 \sinh \omega_2 \cos \Omega_3.$$

If $\Omega_3 = \pi$ this gives

$$\begin{aligned} \cosh \omega_3 &= \cosh \omega_1 \cosh \omega_2 + \sinh \omega_1 \sinh \omega_2 \\ &= \cosh(\omega_1 + \omega_2), \end{aligned}$$

and thus

$$\omega_3 = \omega_1 + \omega_2.$$

It will be seen that the formula giving $\cosh \omega_3$ in terms of ω_1 , ω_2 , and Ω_3 is analogous to the well-known formula in spherical trigonometry, and, in fact, represents the formula connecting three sides and an angle for the case of a triangle on a sphere of radius $\sqrt{-1}$. If we prefer so to express it, it is the formula connecting three sides and an angle in a Lobatschefskij triangle.



We have now to show that similar relations hold in respect to the other particles. Consider a plane drawn perpendicular to the axis of z and meeting it in the point C .

Suppose the plane meets the lines (1) and (2) in A and B respectively and consider the triangle A, B, C .

We have $\hat{C} = \Omega_3$.

Also $\tan A = \frac{a \sin C}{b - a \cos C}$.

Now A is the angle at particle (1) as observed from particle (3), but is not the angle as observed from (1) itself. If Ω_1 be this latter angle we have already seen that

$$\tan \Omega_1 = \frac{\tan A}{\cosh \omega_2}.$$

But $a : b = \tanh \omega_1 : \tanh \omega_2$.

$$\text{Thus } \tan^2 \Omega_1 = \frac{\tanh^2 \omega_1 (1 - \cos^2 C)}{\cosh^2 \omega_2 \{ \tanh \omega_2 - \tanh \omega_1 \cos C \}^2}.$$

$$\text{But } \cos C = \cos \Omega_3 = \frac{\cosh \omega_1 \cosh \omega_2 - \cosh \omega_3}{\sinh \omega_1 \sinh \omega_2}.$$

$$\begin{aligned} \text{Thus } \tan^2 \Omega_1 &= \frac{\tanh^2 \omega_1 \left\{ 1 - \left(\frac{\cosh \omega_1 \cosh \omega_2 - \cosh \omega_3}{\sinh \omega_1 \sinh \omega_2} \right)^2 \right\}}{\cosh^2 \omega_2 \left\{ \tanh \omega_2 - \tanh \omega_1 \left(\frac{\cosh \omega_1 \cosh \omega_2 - \cosh \omega_3}{\sinh \omega_1 \sinh \omega_2} \right) \right\}^2} \\ &= \frac{\sinh^2 \omega_1 \sinh^2 \omega_2 - (\cosh \omega_1 \cosh \omega_2 - \cosh \omega_3)^2}{(\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1)^2} \\ &= \frac{1 - \cosh^2 \omega_1 - \cosh^2 \omega_2 - \cosh^2 \omega_3 + 2 \cosh \omega_1 \cosh \omega_2 \cosh \omega_3}{(\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1)^2}. \end{aligned}$$

$$\begin{aligned} \text{Thus } \frac{1}{\cos^2 \Omega_1} &= \frac{1 - \cosh^2 \omega_2 - \cosh^2 \omega_3 + \cosh^2 \omega_2 \cosh^2 \omega_3}{(\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1)^2} \\ &= \frac{\sinh^2 \omega_2 \sinh^2 \omega_3}{(\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1)^2} \end{aligned}$$

Thus, extracting square roots, we get

$$\cos \Omega_1 = \frac{\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1}{\sinh \omega_2 \sinh \omega_3}$$

$$\text{or } \cosh \omega_1 = \cosh \omega_2 \cosh \omega_3 - \sinh \omega_2 \sinh \omega_3 \cos \Omega_1.$$

By a similar process we may obtain a third formula of the same type so that we see that the relation between the three particles is such that we may regard *any one* of them as "at rest," and the remaining two as in motion with respect to it.

Thus instead of a Euclidian triangle of velocities, we get a Lobatschefskij triangle of rapidities. For small rapidities, however, we may identify rapidity and velocity, and the Lobatschefskij triangle may be treated as a Euclidian one. It is also seen that rapidities in the same straight line are additive.

The formulæ which we have obtained agree with those of Einstein, if we take the "velocity of light" as unity and express the results in terms of velocities instead of rapidities.

They have also been deduced from Minkowski's theory by Sommerfeld.

It will be observed that rapidities may be as great as we please, but velocities must always be less than a certain finite quantity which is equal to unity in the units which we have selected.

OTHER FORMULÆ.

Various other formulæ, analogous to the formulæ of spherical trigonometry, may be obtained connecting the parts of a triangle of rapidities.

Thus we have, for instance,

$$\begin{aligned} \sin^2 \Omega_1 &= 1 - \left(\frac{\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1}{\sinh \omega_2 \sinh \omega_3} \right)^2 \\ &= \frac{(\cosh^2 \omega_2 - 1)(\cosh^2 \omega_3 - 1) - (\cosh \omega_2 \cosh \omega_3 - \cosh \omega_1)^2}{\sinh^2 \omega_2 \sinh^2 \omega_3}, \\ \frac{\sin \Omega_1}{\sinh \omega_1} & \\ &= \frac{\sqrt{\{1 - \cosh^2 \omega_1 - \cosh^2 \omega_2 - \cosh^2 \omega_3 + 2 \cosh \omega_1 \cosh \omega_2 \cosh \omega_3\}}}{\sinh \omega_1 \sinh \omega_2 \sinh \omega_3}. \end{aligned}$$

From the symmetry of the expression on the right it follows that

$$\frac{\sin \Omega_1}{\sinh \omega_1} = \frac{\sin \Omega_2}{\sinh \omega_2} = \frac{\sin \Omega_3}{\sinh \omega_3}.$$

Again it is easy to deduce the formula

$$\cos \Omega_1 = -\cos \Omega_2 \cos \Omega_3 + \sin \Omega_2 \sin \Omega_3 \cosh \omega_1,$$

and two others of the same type.

LOBATSCHESKIJ SYSTEM.

It is interesting to note that a system of particles diverging in all directions with various uniform relative rapidities from simultaneous contact may be regarded as a kind of Lobatscheskij body. Any three such particles, as we have seen, give a Lobatscheskij triangle of rapidities. If we select any one of the particles, the remainder diverge from it in various directions. If we suppose a small Euclidian system of permanent configuration to be associated with the selected particle, to serve as a system of reference, these

directions will be connected by the relations of spherical trigonometry. As is well known, however, spherical trigonometry is common both to Euclidian and Lobatschewskijian geometry, so that the whole system of diverging particles may be regarded as a sort of Lobatschewskij body.

An ordinary Euclidian body may be regarded as a limiting case in which the instant of simultaneous contact is removed to infinity.

PHYSICAL SIGNIFICANCE OF THE INDEX OF A PARTICLE.



In order to obtain a clearer physical conception of the index of a particle which is in motion with respect to our fundamental particle A , let us suppose the latter to be fixed with respect to a plane mirror at one-half the unit distance in front of it.

We shall suppose a second particle P which is initially in contact with A to move parallel to the surface of the mirror with uniform velocity v .

Suppose now we take the instant at which the particles are in contact as that at which both have the index zero, and suppose that at that instant a flash of light goes out from them to the mirror. Then the index of A at the instant of the n^{th} arrival of the light at A is n ; while it is easy to show that the index of P at the instant of the n^{th} arrival of the light at P is also n . In order to show this we imagine a flash of light to go from A to P directly and back again to A .

Let N_a and N_r be the indices of departure and return of the light.

Since the velocity of the particle is supposed constant, we have

$$\frac{N_r - N_a}{2} = v \frac{N_r + N_a}{2}.$$

Also

$$N_r N_a = \left(\frac{N_r + N_a}{2} \right)^2 - \left(\frac{N_r - N_a}{2} \right)^2 = \left(\frac{N_r + N_a}{2} \right)^2 (1 - v^2).$$

Thus

$$\frac{d(N_r N_a)}{d(N_r + N_a)} = \frac{N_r + N_a}{2} (1 - v^2),$$

and

$$2 \frac{d^2(N_r N_a)}{d(N_r + N_a)^2} = 1 - v^2.$$

This gives

$$N_a = \frac{N_r + N_a}{2} \sqrt{1 - v^2}.$$

If now we put $N_a = 1$, we get

$$\frac{N_r + N_a}{2} = \frac{1}{\sqrt{(1-v^2)}}.$$

Thus

$$\frac{N_r - N_a}{2} = \frac{v}{\sqrt{(1-v^2)}}.$$

This is the distance from A of the position of P , when P has the index unity.

The distance travelled by light in going from A to the mirror and from the mirror to this position is

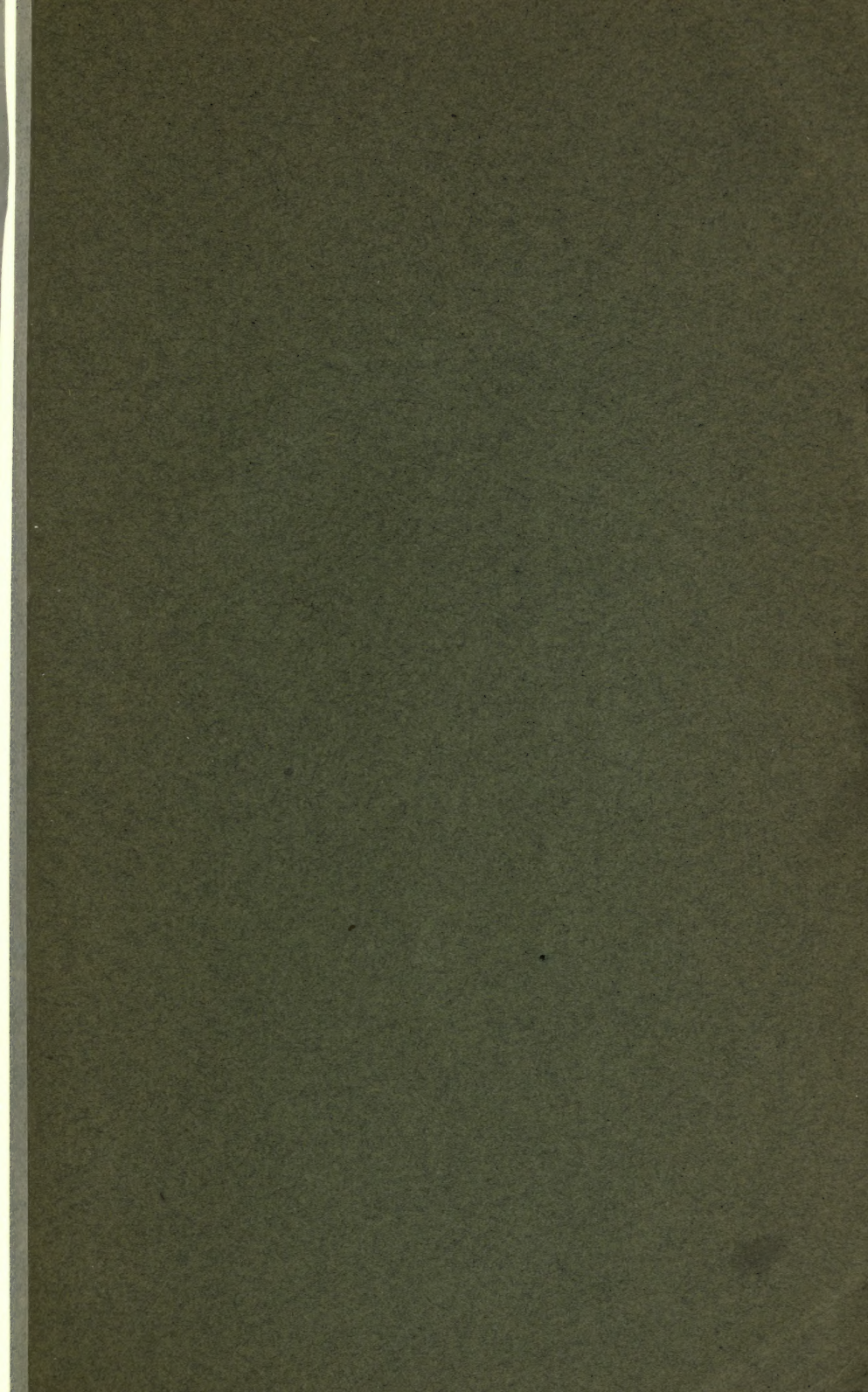
$$2 \sqrt{\left[\left(\frac{1}{2} \right)^2 + \left(\frac{v}{2\sqrt{(1-v^2)}} \right)^2 \right]} = \frac{1}{\sqrt{(1-v^2)}}.$$

But in our system of units the distance travelled by the particle P in the same interval is v times this or $\frac{v}{\sqrt{(1-v^2)}}$, which is the distance from A of the position of P at the instant when P has the index unity. This proves the result stated.

Now, if we have any number of systems of permanent configuration which are moving with respect to one another with uniform velocities in fixed directions and without rotation, we may always imagine one particle of each system such that all such particles are in contact simultaneously. The index of these might be supposed to be given by the mirror method, while the index of any other, moving in the same direction and with the same velocity, might be supposed to be determined by the arithmetic mean theorem.

We may also offer the following *suggestion* as to index, which, *if permissible*, renders its meaning more definitely physical:—

The number of vibrations corresponding to a definite spectrum line of a particular substance, which are executed in any interval, is proportional to the difference of index of the particle emitting the light at the beginning and end of the interval, the constant of proportion being fixed for each particular line. This is on the assumption that the velocities are constant.



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